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ESTIMATION OF FIN EFFICIENCIES OF REGULAR TUBES ARRAYED IN CIRCUMFERENTIAL FINS

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NOMENCLATURE

A	area of symmetry element [m^2]
b	fin thickness [m]
h	convective heat transfer coefficient between fin and surrounding fluid [W m^{-2} K^{-1}]
k	thermal conductivity of fin material [W m^{-1} K^{-1}]
L	characteristic length scale [m]
r_1, r_2	outer tube and outer fin radii of equivalent circular tube and fin system [m]
T_f	temperature of fluid within tube [K]
T_∞	temperature of fluid surrounding fin [K]
U	overall heat transfer coefficient of tube and fluid within [W m^{-2} K^{-1}].

Greek symbols

Γ	tube perimeter [m]
η	fin efficiency
μ	LU/k, dimensionless
ψ	$2hL^2/bk$, dimensionless
Ω	area of fin [m^2].

INTRODUCTION

THE HEAT transfer effectiveness of a tubular heat exchanger is increased by attachment of circumferential fins to the tubes. The heat exchanger could be a single tube and fin, Fig. 1(a), or an array of tubes mounted in a fin, Fig. 1(b). The shapes of the tubes and geometry of the arrays depend on the duty of the heat exchanger. In designing a tube and fin system, fin efficiency is one of the necessary inputs. Except for the simplest systems, fin efficiency must be determined numerically. We have found, however, that the fin efficiency of a great many arrays can be well approximated analytically by an equivalent circular tube and fin heat exchanger. The finding was motivated by our studies of an analogous problem in catalysis [1, 2]. In what follows, we present the equivalent circular fin and tube model and identify those arrays for which it provides an accurate estimate of fin efficiency.

FIN EFFICIENCY

The temperature of a circumferential fin at steady state is determined by the equation

$$k\nabla^2 T - \frac{2h}{b}(T - T_\infty) = 0, \tag{1}$$

where ∇^2 is the two-dimensional Laplacian (defined in the plane of the fin), b and k the fin thickness and thermal conductivity (assumed to be constant), T_∞ the temperature of the fluid surrounding the fin, and h the heat transfer coefficient between the fin and the surrounding fluid. The boundary condition along the outer circumference Γ of the tube is typically

$$k(\nabla T)\cdot\hat{n} = U(T - T_f) \quad \text{on } \Gamma, \tag{2}$$

where $\hat{n}$ is the outward normal to the tube along the base of the fin, U the overall heat transfer coefficient of the tube and fluid within, and T_f the temperature of the fluid within the tube. The outer fin boundary condition for all the systems considered herein is

$$\nabla T\cdot\hat{n} = 0, \tag{3}$$

characteristic of an insulating or symmetry boundary. The heat transfer coefficients h and U are assumed to be constant.

Fin efficiency η can be defined as [3]

$$\eta \equiv \int_\Omega (T - T_\infty)\,dA/\Omega(T_f - T_\infty), \tag{4}$$

where Ω is the area of the fin. An efficiency of unity would result from $k = U = \infty$.

Introducing a characteristic length scale L, we can transform the problem to dimensionless form:

$$\begin{aligned} \nabla^{*2}T - \psi^2(T - T_\infty) &= 0 \quad \text{in } \Omega^*, \\ (\nabla^* T)\cdot\hat{n} &= \mu(T - T_f) \quad \text{on } \Gamma^*, \end{aligned} \tag{5}$$

where ψ and μ are dimensionless parameters

$$\psi^2 = 2hL^2/bk, \quad \mu = LU/k, \tag{6}$$

and $\nabla^* \equiv L\nabla$, $\Omega^* \equiv \Omega/L^2$, and $\Gamma^* \equiv \Gamma/L$. We are free to

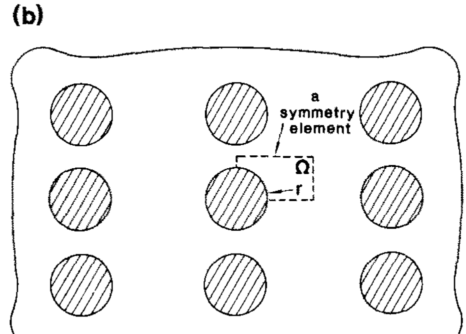


FIG. 1. (a) Circular disk fin of radius r_2 attached to a circular tube of outer radius r_1. (b) Cross section of a square array of circular tubes in a sheet fin.

identify the length L with a pertinent dimension of the heat exchanger. For example, in a circular tube and fin exchanger, L can be taken to be the outer radius of the tube.

Small values of ψ result for sufficiently large fin conductivity. In such a limit the fin temperature is approximately uniform and can be determined from the energy balance

$$2h\Omega(T_\infty - T) = bU\Gamma(T - T_f). \tag{7}$$

This result yields for the fin efficiency the equation

$$\frac{1}{\eta} - 1 = \frac{\psi^2}{\mu}\frac{\Omega}{L\Gamma}, \tag{8}$$

valid for sufficiently small ψ.

The other extreme, large ψ, corresponds to small fin conductivity (compared to $2hL^2/b$) and can also be treated simply. In this case the fin is approximately at the surrounding fluid temperature T_∞ everywhere except in a boundary region close to the tube perimeter Γ. If the boundary region is thin compared to the radius of curvature of the tube, then equation (5) reduces to the one-dimensional problem

$$\frac{d^2T}{dx^{*2}} = \psi^2(T - T_\infty), \quad \left.\frac{dT}{dx^*}\right|_{x^*=0} = \mu(T - T_f)|_{x^*=0}, \qquad T \to T_\infty \quad \text{as} \quad x^* \to \infty, \tag{9}$$

where x is the normal distance away from the tube perimeter Γ. The solution to this equation is $T = [(T_f - T_\infty)\mu/(\psi + \mu)]\,e^{-\psi x^*} + T_\infty$, which yields the formula

$$\eta = \frac{\Gamma L \mu}{\Omega\psi(\psi + \mu)}, \tag{10}$$

for the high ψ fin efficiency. It is, of course, unlikely that one would purposefully design a heat exchanger with large ψ, although it is conceivable that a poor fin situation might be encountered in systems for which other criteria than heat exchange dictate design.

The ranges of parameters ψ over which the asymptotic expressions, equations (8) and (10), are valid depend on tube geometry and array and on the values of μ, Ω, L, and Γ. In general this dependence must be determined by numerical solution of equation (5). In what follows, we give for many arrays ranges of validity of the asymptotic formulas and outline an approximation that allows accurate estimates of fin efficiencies for these arrays outside the asymptotic ranges.

CIRCULAR TUBE AND FIN SYSTEM

Consider the circular tube and fin system shown in Fig. 1. Set $L = r_1$, the outer radius of the tube. $\Omega = \pi(r_2^2 - r_1^2)$ and $\Gamma = 2\pi r_1$, where r_2 is the outer radius of the fin. Equation (5) is for this system

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T}{\partial r^*}\right) - \psi^2(T - T_\infty) = 0, \qquad \left.\frac{\partial T}{\partial r^*}\right|_{r^*=1} = \mu(T - T_f)|_{r^*=1}; \quad \left.\frac{\partial T}{\partial r^*}\right|_{r^*=r_2/r_1} = 0, \tag{11}$$

which has the solution

$$\frac{T - T_f}{T_\infty - T_f} = 1 - \mu M(r)/\{\mu M(r_1) + \psi M(r_1)\}, \tag{12}$$

with

$$M(r) = I_0(\psi r/r_1)K_1(\psi r_2/r_1) + K_0(\psi r/r_1)I_1(\psi r_2/r_1), \qquad N(r) = -I_1(\psi r/r_1)K_1(\psi r_2/r_1) + K_1(\psi r/r_1)I_1(\psi r_2/r_1). \tag{13}$$

I_0, I_1 and K_0, K_1 are the first and second kind of modified Bessel functions of orders zero and one.

The fin efficiency for this system is

$$\eta_c = [2\mu r_1^2/\psi^2(r_2^2 - r_1^2)]\left\{1 + \frac{\mu M(r_1)}{\psi N(r_1)}\right\}^{-1}. \tag{14}$$

The small and large ψ limits of equation (14) agree of course with the general results, equations (8) and (10). More importantly, the exact result provides a means of estimating the domains of validity of the asymptotic results. In particular, if $\psi r_2/r_1 < 0.1$, the fin efficiency is predicted by equation (8) with less than 1% error; and if $\psi > 10$, the fin efficiency is predicted by equation (10) with less than 1% error.

Table 1. Comparison of fin efficiency, η_{FEM}, computed by finite element mathematics, with the fin efficiency, η_c, of the equivalent circular tube and fin

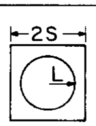
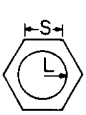
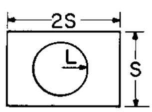
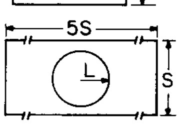
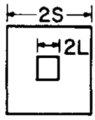
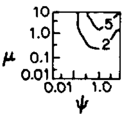
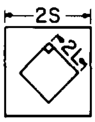
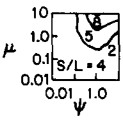
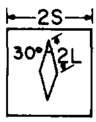
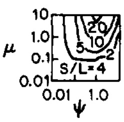
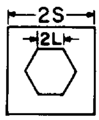
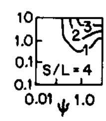
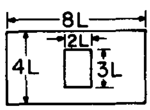
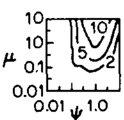
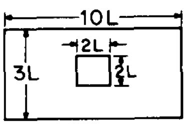
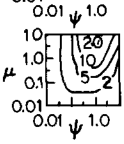
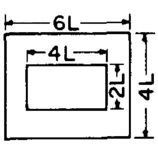
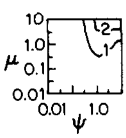
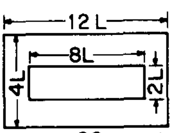
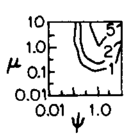
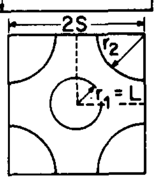
Cross-section of symmetry element	Range of S	Equivalent tube and fin radii r_1	r_2	Percentage error $E \equiv \left\lvert\frac{\eta_{FEM}}{\eta_c} - 1\right\rvert \times 100\%$
(a) 	$1.5 \le S/L \le 10$	L	$\frac{2S}{\sqrt{\pi}}$	$E \le 1.5\%$, $0.01 \le \psi, \mu \le 10$
(b) 	$2 \le S/L \le 10$	L	$\sqrt{\frac{3\sqrt{3}}{2\pi}}\,S$	$E \le 1\%$, $0.01 \le \psi, \mu \le 10$
(c) 	$2.2 \le S/L \le 20$	L	$\sqrt{\frac{2}{\pi}}\,S$	$E \le 4\%$, $\psi = 0.01, 0.1, 10$ $E \le 17\%$, $\psi = 1.0$
(d) 	$3 \le S/L \le 16$	L	$\sqrt{\frac{5}{\pi}}\,S$	$E \le 2\%$, $\psi = 0.01, 10$ $E \le 20\%$, $\psi = 0.1, 1.0$

Table I. (continued)

	Cross-section of symmetry element	Range of S	Equivalent tube and fin radii r_1	r_2	Percentage error $E \equiv \left\lvert \frac{\eta_{FEM}}{\eta_c} - 1 \right\rvert \times 100\%$	
(e)		$1.5 \le S/L \le 4$	L	S		$E(S/L<4) < E(S/L=4)$
(f)		$1.5 \le S/L \le 4$	L	S		$E(S/L<4) < E(S/L=4)$
(g)		$2 \le S/L \le 4$	$L/2$	$S/\sqrt{2}$		$E(S/L<4) < E(S/L=4)$
(h)		$2.5 \le S/L \le 4$	$\sqrt{3}\, L$	$\sqrt{\frac{2}{\sqrt{3}}}\, S$		$E(S/L<4) < E(S/L=4)$
(i)			$1.2L$	$2.77L$		
(j)			L	$2.74L$		
(k)			$1.33L$	$2.31L$		
(l)			$1.6L$	$2.77L$		
(m)		$S/L = 5$ $2 \le r_2/L \le 4$	$\frac{r_1^2 + r_2^2}{r_1 + r_2}$	$\frac{2S}{r_1+r_2} \times \sqrt{\frac{r_1^2+r_2^2}{\pi}}$	$E \le 3\%$, $r_2 = 2, 3, 4$ $0.01 \le \psi$, $\mu \le 10$	

EQUIVALENT CIRCULAR TUBE AND FIN APPROXIMATION

For arrays of tubes, equation (5) usually must be solved numerically even though the problem is reduced to a symmetry element [see Fig. 1(b) and Table 1] for regular arrays. On the basis of our findings for a mathematically similar problem in catalysis, it appears that costly numerical work can be avoided by estimating the fin efficiency of a given array from an 'equivalent' circular tube and circular fin system.

The outer tube radius r_1 and the outer fin radius r_2 of the equivalent circular tube and fin system are defined by

$$r_1 = \frac{2(A-\Omega)}{\Gamma} = \frac{2(\text{cross-section area of tube in symmetry element})}{\text{outer perimeter of tube in symmetry element}}, \tag{15}$$

and

$$\frac{r_2^2 - r_1^2}{2r_1} = \frac{\Omega}{\Gamma} = \frac{\text{area of fin in symmetry element}}{\text{outer perimeter of tube in symmetry element}}, \tag{16a}$$

or

$$r_2 = 2\sqrt{A(A-\Omega)}/\Gamma, \tag{16b}$$

where A is the area of the symmetry element. With these definitions the asymptotic limits (large and small ψ) of equation (14) agree with the general results, equations (8) and (10). The equivalent circular tube and fin approximation is to compute the radii from equations (15) and (16) for a given heat exchanger and to estimate its fin efficiency from the analytical formula, equation (14), for a circular tube and fin system.

We have tested the approximation for several different arrays with values of μ and ψ ranging from 0.01 to 10 (L being set equal to r_1). The test was to compare the fin efficiency η_{FEM} computed from the finite element numerical solution to equation (5) with the fin efficiency η_c predicted by the equivalent circular tube and fin heat exchanger. Our results are summarized in Table 1. The percentage error $E(\equiv |\eta_{FEM}/\eta_c - 1| \times 100\%)$ is indicated in Table 1 either by an error bound for ranges or particular values of μ and ψ [e.g. examples (a) or (b) in Table 1] or as plots of constant error curves in the $\mu-\psi$ plane [e.g. example (e) in Table 1].

It is unlikely that by design a heat exchanger will be chosen for which $\psi > 1$. Thus, it is significant that for $\psi < 1$, the equivalent circular tube and fin approximation is quite good for all but the most elongated symmetry elements [examples (g) and (j) in Table 1].

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2. D.-Y. Kuan, R. Aris and H. T. Davis, Effectiveness of catalytic archipelagos: II. Random arrays of random islands, *Chem. Engng Sci.* (1983).
3. See, e.g. F. Kreith and W. Z. Black, *Basic Heat Transfer*, p. 79. Harper & Row, New York (1980). Note that our definition has used the fluid temperature inside the tube instead of the wall temperature of the tube.